

Final Exam Formula Sheet

- For a regular curve $\vec{r}(t)$,

$$- \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$- \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$$- \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

- For a regular curve $\vec{r}(t)$, the curvature is given by

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$

- **The Second Derivative Test for Functions of Two Variables**

Let $f(x, y)$ be a twice differentiable function, and assume its second partial derivatives are continuous. Let (a, b) be a critical point for f , and define the Hessian of f at (a, b) to be

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

- If $D(a, b) < 0$, then (a, b) is a Saddle Point.
- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a Local Minimum.
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a Local Maximum.
- If $D(a, b) = 0$, the test is inconclusive.

- **Lagrange Multipliers**

If a differentiable function f has a local maximum or local minimum on a constraint curve of the form $g = \text{constant}$ (where g is a differentiable function), then there is a constant λ so that

$$\nabla f = \lambda \nabla g$$

at the local extremum (so long as ∇g is not zero at the point in question).

- **Polar Coordinates**

- $x = r \cos \theta, \quad y = r \sin \theta$
- $r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}$
- $dA = r \, dr \, d\theta$

- **Cylindrical Coordinates**

- $x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$
- $r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}, \quad z = z$
- $dV = r \, dr \, d\theta \, dz$

- **Spherical Coordinates**

- $x = \rho \cos \theta \sin \varphi, \quad y = \rho \sin \theta \sin \varphi, \quad z = \rho \cos \varphi$
- $\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan \theta = \frac{y}{x}, \quad \varphi = \arccos(z/\rho)$
- $dV = \rho^2 \sin \varphi \, d\rho d\theta d\varphi$

- **Change of Variables in Multiple Integrals**

Suppose a transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is one-to-one (except perhaps on the boundary of \mathcal{D}), maps a region \mathcal{D} in the uv -plane to a region \mathcal{C} in the xy -plane, and g and h have continuous partial derivatives in a region containing \mathcal{D} . If f is a continuous function over the region \mathcal{C} , then

$$\iint_{\mathcal{C}} f(x, y) dx dy = \iint_{\mathcal{D}} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where the Jacobian of the transformation is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

The analogous statement holds for triple integrals.

- **Conservative Vector Fields and Path Independence**

If a vector field \vec{F} is defined on a simply connected open domain and the components of \vec{F} have continuous partial derivatives, then \vec{F} is conservative if and only if

$$\nabla \times \vec{F} = \vec{0}$$

(i.e. there is a potential function f so that $\nabla f = \vec{F}$). Line integrals of conservative vector fields are independent of path.

- **Green's Theorem**

Suppose a vector field $\vec{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}$ has continuous partial derivatives on an open set in the plane containing the region \mathcal{D} whose boundary is a positively oriented, piecewise smooth simple closed curve \mathcal{C} . Then the counter-clockwise circulation around \mathcal{C} is given by

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \oint_{\mathcal{C}} P dx + Q dy = \iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy.$$

For the outward flux from \mathcal{C} (again, oriented positively),

$$\oint_{\mathcal{C}} \vec{F} \cdot \hat{n} ds = \oint_{\mathcal{C}} P dy - Q dx = \iint_{\mathcal{D}} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} dx dy.$$

- **Surface Integrals**

Suppose a surface \mathcal{S} is parametrized by $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ over a region \mathcal{D} in the uv -plane. Assuming the partial derivatives exist, the integral of a scalar function f over the surface \mathcal{S} is computed by

$$\iint_{\mathcal{S}} f \, dS = \iint_{\mathcal{D}} f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| \, du \, dv.$$

Moreover, if \mathcal{S} is orientable and the orientation is chosen appropriately, the flux of a vector field \vec{F} through the surface is computed by

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{D}} \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv.$$

- **Stokes' Theorem**

Let \mathcal{S} be a piecewise smooth, oriented surface that is bounded by a simple, closed, piecewise smooth curve $\mathcal{C} = \partial\mathcal{S}$ given the induced orientation from \mathcal{S} . Suppose a vector field \vec{F} has continuous partial derivatives on an open region containing \mathcal{S} . Then

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} (\nabla \times \vec{F}) \cdot d\vec{S}.$$

- **The Divergence Theorem**

Let \mathcal{W} be a simple solid region in space that is bounded by a closed surface $\mathcal{S} = \partial\mathcal{W}$ given the outward orientation. Suppose a vector field \vec{F} has continuous partial derivatives on an open region containing \mathcal{W} . Then

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{W}} (\nabla \cdot \vec{F}) \, dV.$$